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# LAMINAR FORCED CONVECTION IN CONVERGING OR DIVERGING PLANAR SYMMETRIC DUCTS

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**Abstract**—Approximate velocity profiles are developed and a new transformation is introduced to reduce the heat transfer problem for laminar forced convection inside a diverging or converging planar symmetric duct subjected to uniform wall temperature to the standard thermal entrance region problem of Graetz type for flow between two parallel plates.

## NOMENCLATURE

$D_e$, equivalent diameter at $z = 0, 4L$;
$L$, half-spacing of conduit at $z = 0$;
$p$, pressure;
$Pe$, Péclet number;
$Pr$, Prandtl number;
$Q(y,z)$, flow rate defined by equation (10);
$Re$, Reynolds number,

$$\frac{u_m(0)4L}{\nu} \quad \text{or} \quad \frac{u_m(z)4\delta(z)}{\nu};$$

$T(y,z)$, temperature;
$T_0, T_w$, the inlet and the wall temperatures respectively;
$u(y,z)$, the axial velocity component;
$u_m(0), u_m(z)$, the mean axial velocity at $z = 0$ and $z$, respectively;
$v(y,z)$, the normal velocity component;
$y, z$, transverse and axial coordinates;
$\bar{y}, \bar{z}$, dimensionless coordinates defined by equations (15).

Greek symbols
$\alpha$, thermal diffusivity;
$\delta(z)$, half-spacing of the channel at the location $z$;
$\Delta$, dimensionless half-spacing of the channel at $z$, $\dfrac{\delta(z)}{L}$;
$\eta$, $\bar{y}/\Delta(\bar{z})$;
$\theta$, dimensionless temperature, $\dfrac{T - T_w}{T_0 - T_w}$;
$\nu$, kinematic viscosity;
$\rho$, density.

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## INTRODUCTION

NUMEROUS extensions of the original Graetz problem [1] have appeared in the literature for laminar forced convection inside a circular tube or a parallel plate conduit. There are also numerous practical applications in which the cross-sectional area for flow varies with the distance along the direction of flow. Zerkle and Sunderland [2] used approximate velocity profiles and introduced a transformation in order to reduce the problem of forced convection inside a circular tube with varying cross-section to the thermal entry region problem for a circular tube with uniform cross-section. The transformation used by Zerkle and Sunderland is not applicable for a planar symmetric duct of varying cross-section. Therefore, the objective of this work is to develop suitable velocity profiles and a new transformation that will reduce the heat transfer problem for laminar forced convection inside a planar symmetric duct of varying cross-section to the standard thermal entrance region problem for forced convection between two parallel plates.

## ANALYSIS

An incompressible, constant property, Newtonian fluid at a uniform temperature $T_0$ flows in steady, fully developed laminar flow between two thermally insulated parallel plates separated by a distance $2L$ before entering the heat transfer section. At the origin of the axial coordinate, $z = 0$, the fluid enters the heat transfer section with a fully developed velocity profile, $u(y)$. In the heat transfer section, $z > 0$, the cross-sectional area for flow varies monotonically with the distance $z$ in the direction of flow, but retains a planar symmetry about the $z$-axis. Figure 1 illustrates the geometry and the coordinates for a converging channel; one may similarly envisage a diverging channel. It is assumed that the walls of the channel in the heat transfer section, $z > 0$, are maintained at a uniform temperature $T_w$, while the walls of the channel in the region $z < 0$ are kept thermally insulated (i.e. no heat transfer). We neglect viscous energy dissipation and the axial heat conduction in the fluid.

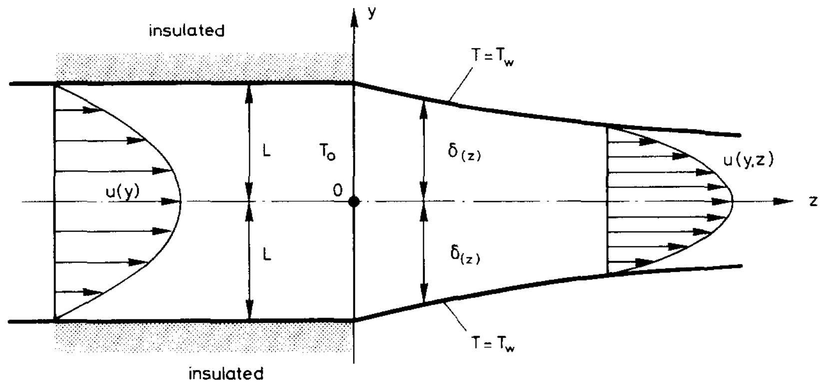


FIG. 1. Geometry and coordinate system.

The governing equations for the 2-dim. flow field are

The continuity equation

$$\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}=0. \tag{1}$$

The momentum equation for the $y$-direction

$$v\frac{\partial v}{\partial y}+u\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right). \tag{2}$$

The momentum equation for the $z$-direction

$$v\frac{\partial u}{\partial y}+u\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right). \tag{3}$$

The boundary conditions for the velocity problem are taken as

$$v(y,z)=0,\ u(y,z)=0 \qquad \text{at} \quad y=\delta(z) \tag{4a, b}$$

$$u(y,z)=\frac{3}{2}u_m(0)\left[1-\left(\frac{y}{L}\right)^2\right] \qquad \text{at} \quad z=0 \tag{4c}$$

$$p(y,z)=p_0 \qquad \text{at} \quad z=0. \tag{4d}$$

Neglecting the axial conduction and the viscous energy dissipation, the temperature distribution in the fluid is governed by the energy equation taken as

$$v\frac{\partial T}{\partial y}+u\frac{\partial T}{\partial z}=\alpha\frac{\partial^2 T(y,z)}{\partial y^2}$$

$$\text{in } 0<y<\delta(z),\ z>0 \tag{5a}$$

and the boundary conditions as

$$\frac{\partial T}{\partial y}=0 \qquad \text{at} \quad y=0,\ z>0, \tag{5b}$$

$$T=T_w \qquad \text{at} \quad y=\delta(z),\ z>0, \tag{5c}$$

$$T=T_0 \qquad \text{at} \quad z=0,\ \text{in } 0\leqslant y\leqslant\delta(z). \tag{5d}$$

If the velocity components are determined from the solution of equations (1)–(4), then the temperature distribution in the fluid is determined from the solution of the energy problem given by equations (5). It is unlikely to obtain an analytic solution to the velocity problem defined by equations (1)–(4). Therefore, we seek an approximate method of analysis for the velocity problem.

It is assumed that the variation of the duct cross-section with the distance along the direction of flow is sufficiently smooth and gradual so that the axial component of the velocity, $u(y,z)$, retains a parabolic profile, but its magnitude changes to satisfy the requirement of continuity. In the case of liquids, this assumption is more valid for the cooling of a liquid in a converging duct or the heating of a liquid in a diverging duct. The reason for this is that the velocity profile becomes flatter for isothermal flow in a converging duct while the viscosity variation during the cooling of a liquid tends to increase the velocity in the central region and decrease it near the walls. As a result, these two opposing effects tend to offset each other for the cooling of a liquid in a converging duct or conversely for the heating of a liquid in a diverging duct.

With these considerations we choose $u(y,z)$ in the form

$$u(y,z)=\frac{3}{2}u_m(z)\left[1-\left(\frac{y}{\delta(z)}\right)^2\right] \tag{6}$$

where the mean axial velocity $u_m(z)$ at the axial location $z$ is related to the mean axial velocity $u_m(0)$ at the origin $z=0$ by the continuity equation as

$$u_m(z)=\frac{L}{\delta(z)}u_m(0). \tag{7}$$

Then the axial velocity profile $u(y,z)$ becomes

$$u(y,z)=\frac{3}{2}u_m(0)\frac{L}{\delta(z)}\left[1-\left(\frac{y}{\delta(z)}\right)^2\right]. \tag{8}$$

To determine the corresponding transverse velocity component $v(y,z)$ we consider the flow rate, $Q(y,z)$, through a cross section area bounded by $y=0$ to $y$ and a unit depth perpendicular to the plane of the figure at the axial location $z$. Then the variation of $Q(y,z)$ across a differential distance d$z$ is related to $v(y,z)$ by

$$v(y,z)=-\frac{\partial Q(y,z)}{\partial z} \tag{9}$$

and $Q(y,z)$ is determined as

$$Q(y,z) = 2\int_0^y u(y',z)\,dy'. \quad (10)$$

Equation (8) is introduced into equation (10) and the integration is performed

$$Q(y,z) = \frac{3}{2}u_m(0)\frac{Ly}{\delta(z)}\left[1 - \frac{1}{3}\left(\frac{y}{\delta(z)}\right)^2\right]. \quad (11)$$

When equation (11) is substituted into equation (9), the transverse velocity component, $v(y,z)$, is determined as

$$v(y,z) = \frac{3}{2}u_m(0)\frac{Ly}{[\delta(z)]^2}\left[1 - \left(\frac{y}{\delta(z)}\right)^2\right]\frac{d\delta(z)}{dz}. \quad (12)$$

Clearly, the velocity components (8) and (9) satisfy both the continuity requirement and the velocity boundary conditions 4(a, b, c).

Now introducing the velocity components given by equations (8) and (12) into the energy equation (5a) we obtain

$$\frac{L}{\delta(z)}\left[1 - \left(\frac{y}{\delta(z)}\right)^2\right]\left\{\frac{3}{2}\frac{y}{\delta(z)}\frac{d\delta(z)}{dz}u_m(0)\frac{\partial T}{\partial y} + \frac{3}{2}u_m(0)\frac{\partial T}{\partial z}\right\} = \alpha\frac{\partial^2 T}{\partial y^2}. \quad (13)$$

This equation is expressed in the dimensionless form as

$$\frac{1}{\Delta}\left[1 - \left(\frac{\bar{y}}{\Delta}\right)^2\right]\left\{\frac{\bar{y}}{\Delta}\frac{d\Delta}{d\bar{z}}\frac{\partial\theta}{\partial y} + \frac{\partial\theta}{\partial\bar{z}}\right\} = \frac{\partial^2\theta}{\partial\bar{y}^2} \quad (14)$$

where the various dimensionless quantities are defined as

$$\left.\begin{aligned} &\bar{y} = \frac{y}{L}, \quad && \bar{z} = \left(\frac{32}{3}\frac{z}{D_e}\right)\Big/ Pe \\ &\Delta = \frac{\delta(z)}{L}, && \theta = \frac{T - T_w}{T_0 - T_w} \\ &\text{and} \\ &D_e = 4L, \quad Pe = Re\,Pr, \quad \frac{y}{\delta} = \frac{\bar{y}}{\Delta}. \end{aligned}\right\} \quad (15)$$

Equation (14) is still too complicated to solve analytically. We wish to introduce a transformation that will remove the function $\Delta$ from equation (14). The following transformation is defined:

$$\eta(\bar{z}) = \frac{\bar{y}}{\Delta(\bar{z})}, \quad \xi(\bar{z}) = \int_0^{\bar{z}}\frac{1}{\Delta(z)}dz. \quad (16a,b)$$

Then various derivatives appearing in equation (14) are determined by the chain rule of differentiation as

$$\frac{\partial}{\partial\bar{y}} = \frac{1}{\Delta}\frac{\partial}{\partial\eta}, \quad \frac{\partial^2}{\partial\bar{y}^2} = \frac{1}{\Delta^2}\frac{\partial^2}{\partial\eta^2} \quad (16c,d)$$

and

$$\frac{\partial}{\partial\bar{z}} = -\frac{\bar{y}}{\Delta^2}\frac{d\Delta}{d\bar{z}}\frac{\partial}{\partial\eta} + \frac{1}{\Delta}\frac{\partial}{\partial\xi}. \quad (16e)$$

Under the transformation (16), the energy equation (14) simplifies to

$$(1 - \eta^2)\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\eta^2}, \text{ in } 0 < \eta < 1, \ \xi > 0 \quad (17a)$$

subject to the boundary conditions

$$\frac{\partial\theta}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0, \ \xi > 0, \quad (17b)$$

$$\theta = 0 \quad \text{at} \quad \eta = 1, \ \xi > 0, \quad (17c)$$

$$\theta = 1 \quad \text{at} \quad \xi = 0, \text{ in } 0 \leqslant \eta \leqslant 1. \quad (17d)$$

The temperature problem defined by equations (17) is the standard Graetz type thermal entrance region problem for flow between two parallel plates and its solution is available [3, 4]. Knowing the temperature distribution, the local Nusselt number is determined as a function of the dimensionless axial variable $\xi$. The relation between $\xi$ and the axial variable $\bar{z}$ is determined according to equation (16b).

The foregoing analysis is valid for both converging and diverging ducts provided that the duct profile varies smoothly with the distance in the direction of flow. In the case of convergent ducts, the reduction in the duct spacing, $\delta(z)$, does not give rise to an increase in the Reynolds number, hence there will be no eventual change of the flow regime from laminar to turbulent. To illustrate this matter, let $Q_t = u_m(z)\delta(z)$ be the flow rate per unit depth at axial location $z$ and the Reynolds number defined as $Re = [u_m(z)4\delta(z)]/\nu$. Then we have $Re = 4Q/\nu$, which implies that the Reynolds number remains the same everywhere along the duct.

## REFERENCES


1. L. Graetz, Über die Wärmeleitungsfähigkeit von Flüssigkeiten, *Ann. Phys. Chem.* **25**, 337–357 (1885).
2. R. D. Zerkle and J. E. Sunderland, The effect of liquid solidification in a tube upon laminar-flow heat transfer and pressure drop, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **90**, 183–189 (1968).
3. J. R. Sellers, M. Tribus and J. S. Klein, Heat transfer in laminar flow in a round tube or flat conduit—The Graetz problem extended, *Trans. Am. Soc. Mech. Engrs* **78**, 441–448 (1956).
4. R. K. Shah and A. L. London, *Laminar Flow Forced Convection in Ducts*. Academic Press, New York (1978).

## CONVECTION FORCEE LAMINAIRE DANS DES CONDUITES CONVERGENTES OU DIVERGENTES A SYMETRIE PLANE

**Résumé**—Des profils de vitesse approchés sont obtenus et une mouvelle transformation est introduite pour résoudre le problème du transfert thermique en convection forcée laminaire dans une conduite divergente ou convergente à symétrie plane, soumise à une température de paroi uniforme pour le problème de la région d'entrée thermique du type de Graetz pour l'écoulement entre deux plans parallèles.

## LAMINARE ERZWUNGENE KONVEKTION IN KONVERGENTEN ODER DIVERGENTEN EBENEN SYMMETRISCHEN KANÄLEN

**Zusammenfassung**—Es werden Näherungen für die Geschwindigkeitsprofile entwickelt und eine neue Transformation eingeführt, mit deren Hilfe das Wärmeübergangsproblem für laminare erzwungene Konvektion in einem divergenten oder konvergenten ebenen symmetrischen Kanal bei einheitlicher Wandtemperatur auf das bekannte Problem der thermischen Einlaufströmung nach Graetz für die Strömung zwischen zwei parallelen Platten zurückgeführt werden kann.

## ЛАМИНАРНАЯ ВЫНУЖДЕННАЯ КОНВЕКЦИЯ В СХОДЯЩИХСЯ ИЛИ РАСХОДЯЩИХСЯ ПЛОСКИХ СИММЕТРИЧНЫХ КАНАЛАХ

**Аннотация** — Получены приближенные профили скорости и предложено новое преобразование для приведения задачи теплопереноса при ламинарной вынужденной конвекции в расходящемся или сходящемся плоском симметричном канале с однородной температурой стенки к обычной задаче теплообмена типа Гретца для течения между двумя параллельными пластинами.